Calculating the average value of the reduced chi squared across a map.

Giles, July 17, 2008

References used are :

BR: Bevington and Robinson, Data Reduction and Error Analysis for the Physical Sciences, Second Edition

CRC: Standard Mathematical Tables, 28th Edition

The probability for obtaining a given value of the chi-squared is:

$$P(x^2;\nu) = \frac{1}{2^{\nu/2}\Gamma(\nu/2)}(x^2)^{(\nu-2)/2}e^{-x^2/2}$$

BR, Appendix C.4

where ν is the number of degrees of freedom.

If one obtains a large number of measurements of this quantity, all obtained for the same value of ν, and then takes an average, the result is:

$$\left\langle x^2 \right\rangle = \frac{\int_0^\infty \frac{1}{2^{\nu/2}\Gamma(\nu/2)}(x^2)^{(\nu-2)/2}e^{-x^2/2}x^2 d(x^2)}{\int_0^\infty \frac{1}{2^{\nu/2}\Gamma(\nu/2)}(x^2)^{(\nu-2)/2}e^{-x^2/2}d(x^2)}$$

Setting u = ($x^2/2$), one gets:

$$\left\langle x^2 \right\rangle = \frac{2^{(\nu-2)/2} \cdot 2 \cdot 2 \int_0^\infty (u)^{(\nu-2)/2} e^{-u} \cdot u \cdot du}{2^{(\nu-2)/2} \cdot 2 \int_0^\infty (u)^{(\nu-2)/2} e^{-u} du}$$

which simplifies to:

$$\left\langle x^2 \right\rangle = \frac{2 \int_0^\infty (u)^{\nu/2} e^{-u} du}{\int_0^\infty (u)^{(\nu/2-1)} e^{-u} du} = 2 \frac{\Gamma(\frac{\nu}{2}+1)}{\Gamma(\frac{\nu}{2})}$$

where the definition of the gamma function comes from equation 597 of CRC.

But since...

$$\Gamma(n+1) = n\Gamma(n) \qquad \text{(equation 601, CRC)}$$

...we can write:

$$\left\langle x^2 \right\rangle = 2 \frac{\frac{\nu}{2}\Gamma(\frac{\nu}{2})}{\Gamma(\frac{\nu}{2})} = \nu$$

Then dividing by the number of degrees of freedom, we see that the average value of the reduced chi squared is unity, regardless of the number of degrees of freedom.