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### Allan Variance

The goal is to calculate the Allan variance in pixel  $(x, y)$  from a set of files taken at times  $t$  with signal  $s(x, y, t)$ . The total number of files is  $N$ .

The average signal at time  $t$  within a width of  $n$  time samples about  $t$  is

$$\langle s(x, y, t) \rangle_n = \frac{1}{n} \sum_{t'=1}^N s(x, y, t') k_n(t - t') \quad (1)$$

where the kernel  $k_n(t)$  is a “top-hat” function of width  $n$  time samples. That is  $k_n(t) = 1$  within  $n$  time samples of its center but  $k_n(t) = 0$  elsewhere. The normalization factor  $n$  is just the total number of files within the kernel width:

$$n = \sum_{t=1}^N k_n(t).$$

To avoid “edge-effects” at the beginning and end of the time series we must also force  $\langle s(x, y, t) \rangle_n = 0$  at the endpoints where  $t - n/2 \leq 0$  and  $t + n/2 > N$ .

To calculate the variance we also need

$$\langle s^2(x, y, t) \rangle_n = \frac{1}{n} \sum_{t'=1}^N s^2(x, y, t') k_n(t - t') \quad (2)$$

We define the Allan variance at each time  $t$  and kernel width  $n$  as

$$\sigma_n^2(t) \equiv \langle (s(t) - \bar{s}_n(t))^2 \rangle = \frac{1}{n} \sum_{t'=t-n/2}^{t'+n/2} [s(t') - \bar{s}_n(t)]^2 \quad (3)$$

$$= \langle s^2(t) \rangle_n - \langle s(t) \rangle_n^2 \quad (4)$$

where we have dropped the  $(x, y)$  pixel dependence for simplicity. The 2 terms in equation (4) are given by equation (2) and the square of equation (1).

The average Allan variance over all times  $t$  within kernel width  $n$  is then

$$\langle \sigma_n^2 \rangle = \frac{1}{N - n + 1} \sum_{t=1}^N \sigma_n^2(t). \quad (5)$$

The normalization factor comes from the fact that there are only  $N - n + 1$  non-zero values of  $\sigma_n^2(t)$  for data with kernel width  $n$  samples.

To calculate the uncertainty on this value at each point  $n$  we have the standard deviation of  $\sigma_n^2(t)$ :

$$\delta(\langle\sigma_n^2\rangle) = \left\{ \frac{1}{N-n} \sum_{t=1}^N [\sigma_n^2(t) - \langle\sigma_n^2\rangle]^2 \right\}^{1/2}. \quad (6)$$

We note that when calculating the Allan variance we could also take into account the uncertainties within each single file, rather than just the file-to-file variations. Without careful derivation here, I will simply state that the answer is:

$$\sigma_n^2(t) \equiv \langle (s(t) \pm \sigma(t) - \bar{s}(t))^2 \rangle_n \quad (7)$$

$$= \langle s^2(t) \rangle_n + \langle \sigma^2(t) \rangle_n - \langle s(t) \rangle_n^2 \quad (8)$$

where, similar to equation (2),

$$\langle \sigma^2(x, y, t) \rangle_n = \frac{1}{n} \sum_{t'=1}^N \sigma^2(x, y, t') k_n(t - t'). \quad (9)$$

As an example of calculating these quantities we use the 110 SHARP data files taken on the source NGC1333. The results for the center pixel of the map are shown in the top of Figure 1.

### $\chi$ -Squared

We define a reduced- $\chi^2$  as

$$\chi_n^2 = \frac{1}{N-n+1} \sum_{t=1}^N \frac{\sigma_n^2(t)}{\langle \sigma^2(t) \rangle_n} \quad (10)$$

where the numerator of the sum is given by equation (7) and the denominator by equation (9). An example of this for the center pixel of NGC1333 is shown in the bottom of Figure 1.

The mean reduced- $\chi^2$  averaged over all pixels in the maps is shown in Figure 2.

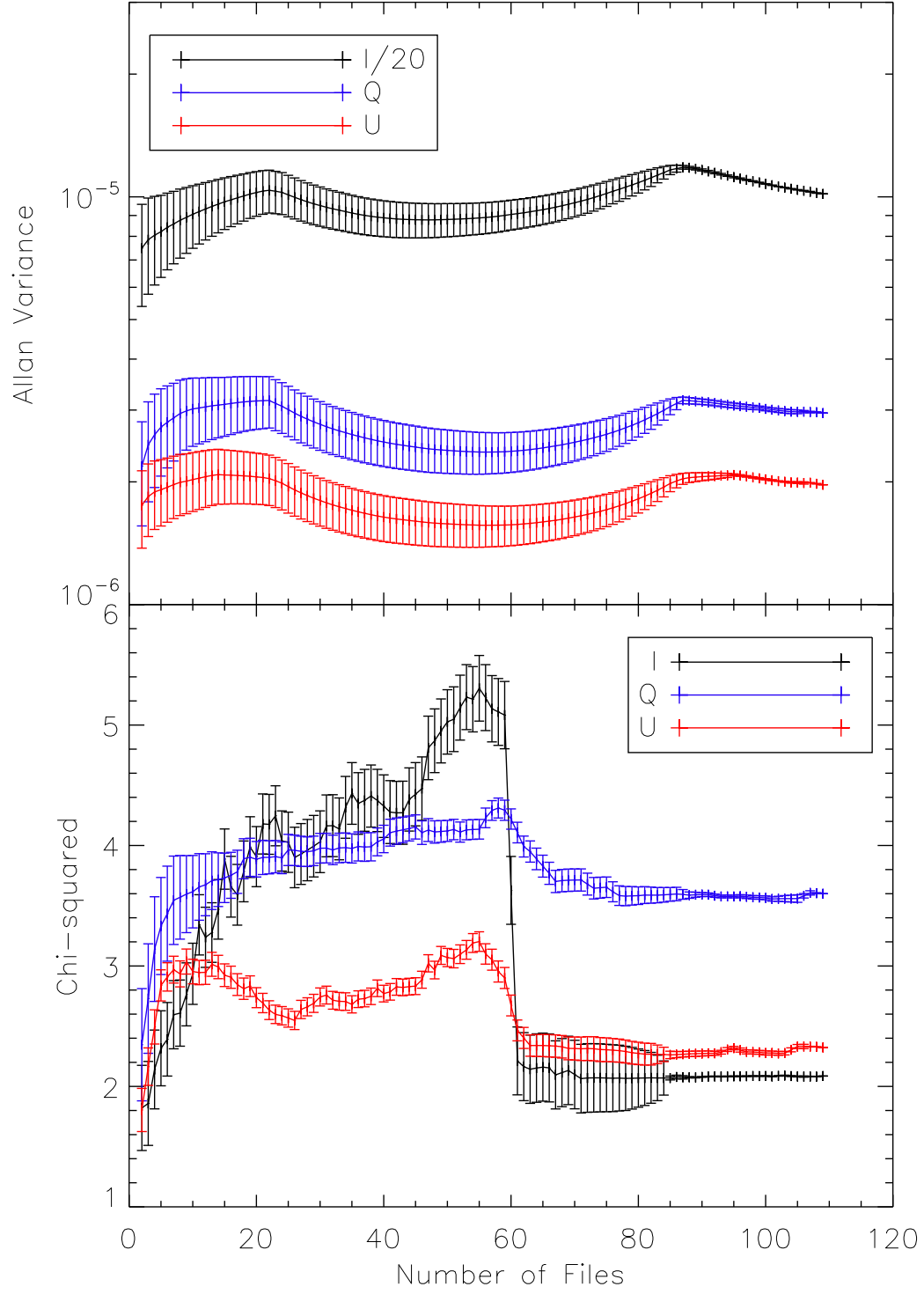


Fig. 1.— *Top*: Allan variance of the center pixel. The  $I$  Stokes parameter has been scaled down by a factor of 20 to fit on the plot. *Bottom*: Mean  $\chi^2$  of center pixel. Error bars represent the standard-deviation-of-the-mean  $\delta(\langle\sigma_n^2\rangle)/\sqrt{N-n}$ .

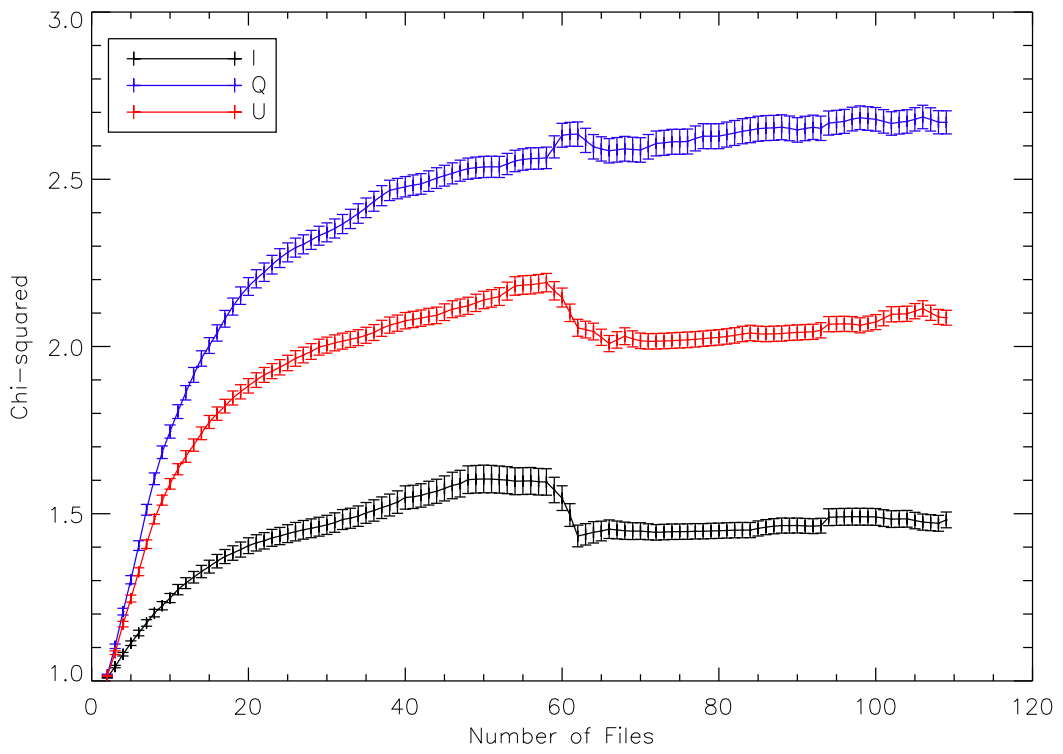


Fig. 2.— Reduced- $\chi^2$  averaged over all  $(x, y)$  pixels in map. Note that this does avoid empty pixels.