Smoothing and Re-sampling

1.1 The Evenly Sampled Grid

I'll start by using some of the same functions John defined in his document. That is,

$$s'(x) = s(x) * b(x), \tag{1}$$

where $s(x), b(x)$, and $s'(x)$ are the source signal, the point spread function (or transfer function), and the "diffraction limited" signal, respectively. The measured signal $t'(x)$ will consist of an evenly sampled version of a pixel-integrated version of $s'(x)$. That is,

$$\begin{aligned}
t'(x) &= \sum_{n=-\infty}^{\infty} \delta(x-nl) \int_{nl-\Delta l/2}^{nl+\Delta l/2} s'(z)\,dz \cdot \\
&= \sum_{n=-\infty}^{\infty} \delta(x-nl) \int_{-\infty}^{\infty} s'(z) \left\{ H\left[z - \left(nl - \frac{\Delta l}{2} \right) \right] - H\left[z - \left(nl + \frac{\Delta l}{2} \right) \right] \right\} dz \\
&= \sum_{n=-\infty}^{\infty} \delta(x-nl) \int_{-\infty}^{\infty} s'(z) \operatorname{rect}\left(\frac{z-nl}{\Delta l} \right) dz \\
&= \sum_{n=-\infty}^{\infty} \delta(x-nl) \int_{-\infty}^{\infty} s'(z) \operatorname{rect}\left(\frac{z-x}{\Delta l} \right) dz \\
&= \left[s'(x) * \operatorname{rect}\left(\frac{x}{\Delta l} \right) \right] \cdot \sum_{n=-\infty}^{\infty} \delta(x-nl),
\end{aligned} \tag{2}$$

where the Heaviside distribution is

$$H(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases} \tag{3}$$

and the rectangular window of width Δl and centered at $x = nl$ is defined as

$$\operatorname{rect}\left(\frac{x-nl}{\Delta l} \right) = \begin{cases} 1, & |x-nl| < \Delta l/2 \\ 0, & |x-nl| > \Delta l/2. \end{cases} \tag{4}$$

For our purposes, Δl is the size of the pixel and l is the spacing between pixels. We can express the measured sampled signal $t'(x)$ as

$$t'(x) = t(x) \cdot \sum_{i=-\infty}^{\infty} \delta(x-il), \tag{5}$$

and if we further define

$$p(x) = \operatorname{rect}\left(\frac{x}{\Delta l}\right), \tag{6}$$

then we can see from equations (1), (2), and (5) that

$$t(x) = \left[s(x) * b(x)\right] * p(x). \tag{7}$$

Note that the presence of the square brackets in equation (7) is not really needed, but is only to emphasize the sequence of transformations on the signal.

We now define the Fourier transform pair relating a function $y(x)$ and its transform $Y(u)$, with u the "frequency" associated to x, as

$$\begin{aligned} y(x) &= \int_{-\infty}^{\infty} Y(u) e^{j2\pi ux} du \\ Y(u) &= \int_{-\infty}^{\infty} y(x) e^{-j2\pi ux} dx. \end{aligned} \tag{8}$$

A periodic function $f(x)$ of period l can also be expressed with the so-called Fourier series

$$f(x) = \sum_{n=-\infty}^{\infty} F(n) e^{j\frac{2\pi nx}{l}}, \tag{9}$$

where the Fourier coefficients are defined with

$$F(n) = \frac{1}{l}\int_{-l/2}^{l/2} f(x) e^{-j\frac{2\pi nx}{l}} dx. \tag{10}$$

Using equations (9) and (10) it is easy to show that

$$\sum_{i=-\infty}^{\infty} \delta(x - il) = \frac{1}{l}\sum_{n=-\infty}^{\infty} e^{j\frac{2\pi nx}{l}}. \tag{11}$$

But since the Fourier transform of a Dirac function $\delta(x)$ equals unity, and the duality of the Fourier transform pair tells us that

$$\int_{-\infty}^{\infty} e^{-j2\pi ux} dx = \delta(-u) = \delta(u), \tag{12}$$

then the combination of equations (5), (11), and (12), gives

$$
\begin{aligned}
T'(u) &= T(u) * \left[\frac{1}{l} \sum_{n=-\infty}^{\infty} \delta\left(u - \frac{n}{l} \right) \right] \\
&= \frac{1}{l} \sum_{n=-\infty}^{\infty} T\left(u - \frac{n}{l} \right),
\end{aligned}
\tag{13}
$$

where

$$
T(u) = B(u) S(u) P(u). \tag{14}
$$

The Fourier transform $P(u)$ due to the pixel integration process is

$$
P(u) = \Delta l \,\mathrm{sinc}(\pi u \Delta l), \tag{15}
$$

where $\mathrm{sinc}(x) \equiv \sin(x)/x$. In the case of SHARP we have $\Delta l \simeq l$.

For the moment, I will continue to consider an infinite number of samples, as in equation (5). In reality, a SHARP image is of finite dimensions, so we should truncate the corresponding summations for the grid, which is the same as convolving the spectrum with a corresponding sinc function.

Assuming that we are dealing with an evenly sampled grid (ESG) of data as in equation (5), we might want to generate a new interpolated grid from this one. Because of the nature of the grid, then the interpolation process is similar to a convolution of this initial grid with the weighting function, followed by a re-sampling of the data. That is, if $w(x)$ is the normalized weighting function, then the new interpolated grid $t_2(x)$ is

$$
\begin{aligned}
t_2(x) &= \sum_{m=-\infty}^{\infty} \delta\left(x - ml - \frac{l}{q} \right) \cdot \left[t'(x) * w(x) \right] \\
&= \sum_{m=-\infty}^{\infty} \delta\left(x - ml - \frac{l}{q} \right) \cdot \left\{ \left[t(x) \cdot \sum_{i=-\infty}^{\infty} \delta(x - il) \right] * w(x) \right\} \\
&= \sum_{m=-\infty}^{\infty} \delta\left(x - ml - \frac{l}{q} \right) \cdot \left[\sum_{i=-\infty}^{\infty} t(il) w(x - il) \right],
\end{aligned}
\tag{16}
$$

where the parameter q determines the displacement of the new grid in relation to the original one. For example, $q = 2$ in the case where the interpolated points are positioned equidistantly from the two neighboring points. Calculating the Fourier transform of the second of equations (16) we get

$$
T_2(u) = \left[\frac{1}{l} \sum_{m=-\infty}^{\infty} \delta\left(u - \frac{m}{l} \right) e^{-j2\pi \frac{m}{q}} \right] * \left[W(u) \cdot \frac{1}{l} \sum_{n=-\infty}^{\infty} T\left(u - \frac{n}{l} \right) \right]. \tag{17}
$$

However, an appropriate choice of the weighting function will ensure that the higher frequency components of $T(u)$ (i.e., when $n \neq 0$) will be filtered out (of course, this assumes that we have at least Nyquist sampling). We can then simplify equation (17) to

$$\begin{aligned} T_2(u) &= \left[\frac{1}{l}\sum_{m=-\infty}^{\infty}\delta\left(u-\frac{m}{l}\right)e^{-j2\pi\frac{m}{q}}\right] * \left[\frac{1}{l}W(u)T(u)\right] \\ &= \frac{1}{l^2}\sum_{m=-\infty}^{\infty}W\left(u-\frac{m}{l}\right)T\left(u-\frac{m}{l}\right)e^{-j2\pi\frac{m}{q}}. \end{aligned} \tag{18}$$

We also write down a new expression for the original sampled grid, which has now also been convoluted with the weighting function and re-sampled (at the original sampling positions). We name this grid $t_1(x)$, and its Fourier transform is

$$\begin{aligned} T_1(u) &= \left[\frac{1}{l}\sum_{m=-\infty}^{\infty}\delta\left(u-\frac{m}{l}\right)\right] * \left[\frac{1}{l}W(u)T(u)\right] \\ &= \frac{1}{l^2}\sum_{m=-\infty}^{\infty}W\left(u-\frac{m}{l}\right)T\left(u-\frac{m}{l}\right). \end{aligned} \tag{19}$$

The fully re-sampled grid $t_3(x)$ is a combination of these two grids with

$$\begin{aligned} T_3(u) &= T_1(u) + T_2(u) \\ &= \frac{1}{l^2}\sum_{m=-\infty}^{\infty}\left\{W\left(u-\frac{m}{l}\right)T\left(u-\frac{m}{l}\right)\left[1+e^{-j2\pi\frac{m}{q}}\right]\right\}. \end{aligned} \tag{20}$$

In the special case where $q = 2$, this equation simplifies to

$$\begin{aligned} T_3(u) &= \frac{1}{l^2}\sum_{m=-\infty}^{\infty}W\left(u-\frac{m}{l}\right)T\left(u-\frac{m}{l}\right)\left[1+e^{-j\pi m}\right] \\ &= \frac{1}{l^2}\left\{\sum_{m=-\infty}^{\infty}W\left(u-\frac{m}{l}\right)T\left(u-\frac{m}{l}\right)\left[1+(-1)^m\right]\right\} \\ &= \frac{2}{l^2}\sum_{m=-\infty}^{\infty}W\left(u-\frac{2m}{l}\right)T\left(u-\frac{2m}{l}\right), \end{aligned} \tag{21}$$

or alternatively

$$t_3(x) = \left[t(x) * w(x)\right] \cdot \frac{1}{l}\sum_{i=-\infty}^{\infty}\delta\left(x - i\frac{l}{2}\right). \tag{22}$$

It is interesting to compare this with the corresponding equation for a hypothetically fully ESG (i.e., with a step of $l/2$) $t''(x)$ that could be obtained from $t(x)$

$$
\begin{aligned}
t''(x) &= t(x) \cdot \sum_{i=-\infty}^{\infty} \delta\left(x - i\frac{l}{2}\right) \\
T''(u) &= \frac{2}{l} \sum_{n=-\infty}^{\infty} T\left(u - \frac{2n}{l}\right).
\end{aligned}
\tag{23}
$$

We see that the only difference between $t_3(x)$ and $t''(x)$, besides the overall scaling factor, is the convolution with the weighting function $w(x)$ for $t_3(x)$. It is, therefore, apparent that $w(x)$ can serve not only as a weighting function, but also as a smoothing function, as in this respect its effect is functionally the same as that of $b(x)$ or any other function that could be applied to $t'(x)$ before the re-sampling process described by equations (16) to (18).

1.1.1 Selection of the weighting function

It seems to me that there are three criterions that must be met for the selection of $w(x)$:

1. It must be such that its Fourier transform filters out frequencies for which $|u| > 1/2l$ (see equation (17)).
2. It must be sufficiently broad so that its amplitude is high enough at the position of the interpolated points, i.e., $w(l/2)$, to ensure an adequate interpolation.
3. Finally, it cannot be too broad so as to affect too much the resolution of the map. That is, the bandwidth of $W(u)$ is constrained by that of $B(u)P(u)$.

To see how we can see our way through this, we will first assume that we satisfy the Nyquist sampling criterion. This implies that the beam width is significantly larger than the pixel size l (by at least a factor of approximately two), and that correspondingly the frequency extent of $B(u)$ is much less than that of $P(u)$. We will therefore neglect the presence of $P(u)$ for the purpose of this discussion, as it would be of little consequence in determining the characteristics of the weighting function under such circumstances. Furthermore, we approximate $b(x)$ by a Gaussian function of width σ

$$
\begin{aligned}
b(x) &= \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} \\
B(u) &= e^{-2\pi\sigma^2 u^2} \equiv e^{-\frac{u^2}{2r^2}}.
\end{aligned}
\tag{24}
$$

For SHARP, the beam size of approximately 9 arcseconds is usually identified with the full-width-half-magnitude (FWHM). This gives $\sigma = \mathrm{FWHM}\big/\sqrt{8\ln(2)} \simeq 3.8$ arcseconds.

I will use standard deviations to specify the widths of Gaussian functions. With this definition, the one-sided bandwidth associated with the SHARP beam is

$$r = \frac{1}{2\sqrt{\pi}\sigma} \simeq \frac{1}{13.5}\ \text{arcseconds}^{-1}. \tag{25}$$

Since $l \simeq 4.7$ arcseconds, it is clear that we are within the Nyquist sampling criterion, as $r^{-1} \gtrsim 2l$. If we were to choose $w(x)$ to also be Gaussian, and of width ϖ, then to satisfy Criterion 1 we must have

$$\varpi \gtrsim \frac{l}{\sqrt{\pi}} \simeq \frac{l}{1.8}. \tag{26}$$

We must also ensure that $4\pi\left(\sigma^2 + \varpi^2\right)$ is not much greater than r^{-2} (from Criterion 3 above). If we try $\varpi = l/\sqrt{\pi}$, then the spatial resolution associated with this choice is

$$\sqrt{\sigma^2 + \varpi^2} = \sigma\sqrt{1 + \frac{l^2}{\pi\sigma^2}} \simeq 1.22\sigma = 4.6\ \text{arcseconds}. \tag{27}$$

This corresponds to an equivalent beam width of 10.9 arcseconds. Finally, the relative amplitude of the weighting function at an interpolated point is

$$\sqrt{2\pi}\varpi\, w\left(\frac{l}{2}\right) = e^{-\frac{\pi}{8}} = 0.68, \tag{28}$$

and the second criterion is equally satisfied.

1.2 The Regularly Sampled Grid

What is a regularly sampled grid (RSG)? I define such a grid as a generalization of the ESG discussed earlier. It is a grid that has the same periodicity as the ESG, but for which the pattern of Dirac functions is more complex. For example, if for the ESG considered above there is one Dirac function within a region of length l, a RSG may have, say, $m > 1$ Dirac functions over the same region. But importantly, the spacing between any two adjacent Dirac functions does not need to be the same everywhere over the length l. See Figure 1.

This description of the regularly sampled grid suggests the following definition: *a regularly sampled grid is one for which the formula for evaluating every interpolated point at a well-defined position within any region of length* l *(one period), as defined with*

$$z'(x)=\frac{\sum_{i=1}^{m} z(x_i)\,w'(x-x_i)}{\sum_{i=1}^{m} w'(x-x_i)}, \tag{29}$$

will all have the same normalization factor $\sum_{i=1}^{m} w'(x-x_i)$.

The reason for suggesting this definition is that existence of a common normalization factor ensures that the interpolation process can be modeled in the same manner as that for the ESG: a convolution followed by a re-sampling. If this condition for a common normalization factor is not met, then we *cannot* model the first step of the re-sampling process with a convolution. This is because the actual weighting function

$$w(x)\equiv\frac{w'(x)}{\sum_{i=1}^{m} w'(x-x_i)} \tag{30}$$

is not the same everywhere (more precisely, its amplitude is position dependent). This will bring us to a discussion of the *irregularly sampled grid* later on. But before we get there, let's verify that a RSG can be treated in the same manner as an ESG.

It should be clear that the RSG is simply a summation of a set of relatively displaced ESG. That is, in this case we can express a RSG $t'(x)$ with

$$t'(x)=t(x)\cdot\sum_{k=1}^{m}\left[\sum_{i=-\infty}^{\infty}\delta\left(x-il-\frac{l}{q_k}\right)\right], \tag{31}$$

where the parameters q_k specifies the relative displacement of each component grid. The Fourier transform is easily calculated to be

$$\begin{aligned} T'(u) &= T(u)*\sum_{k=1}^{m}\left[\frac{1}{l}\sum_{n=-\infty}^{\infty}\delta\left(u-\frac{n}{l}\right)e^{-j2\pi\frac{n}{q_k}}\right] \\ &= \frac{1}{l}\sum_{k=1}^{m}\left[\sum_{n=-\infty}^{\infty}T\left(u-\frac{n}{l}\right)e^{-j2\pi\frac{n}{q_k}}\right]. \end{aligned} \tag{32}$$

Now, if the weighting function is such as to meet the Criterion 1 on page 5, then convolving $t'(x)$ with it will filter out the higher frequency part of the spectrum (i.e., where $n\neq 0$) and the interpolated grid is expressed with

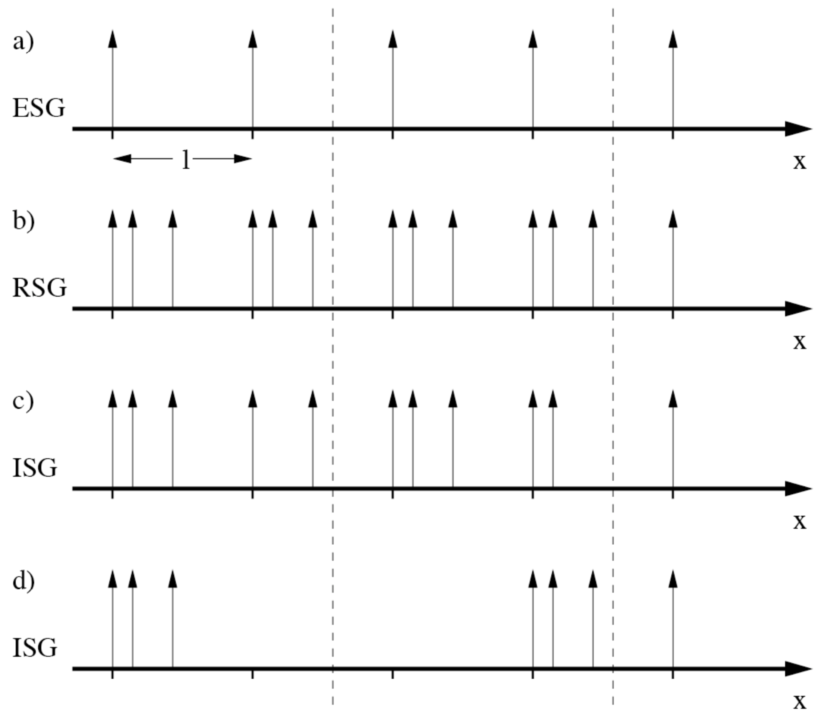

Figure 1 – Examples of a) an ESG, b) a RSG, and c) and d) two ISG are shown. Interpolations at the positions of the two vertical broken lines would require weighting function that have a common normalization factor for the ESG and RSG, and different normalization factors for the ISG.

$$T_2(u) = \frac{m}{l^2} \sum_{n=-\infty}^{\infty} W\left(u - \frac{n}{l}\right) T\left(u - \frac{n}{l}\right) e^{-j2\pi\frac{n}{q}}, \tag{33}$$

where, as before, the parameter q determines the displacement of the new grid in relation to the original one. The grid $t_2(x)$ is an ESG that was generated from the RSG $t'(x)$. Obviously, we could proceed as before with another grid $t_1(x)$, and complete the re-sampling process.

1.3 The Irregularly Sampled Grid

Figure 1 c) and d) show two examples of irregularly sampled grids (ISG). The first example is a case where the Dirac functions may fall at different position within successive intervals of period l, while the second is a case where no sampling occurs for some periods (otherwise the Dirac functions form a regular pattern when present). The last example can be thought as being similar to the case of an imperfect detector array where some pixels are missing (as is the case for SHARP). At this point I have not found some general way of treating examples such as these. The problem with these ISG is twofold:

1. There is no obvious way of expressing the Fourier transform of an irregularly spaced Dirac train so that the spectrum will still show a repeating pattern of some frequency (which would be a function of the inverse of the "mean" sampling period, such as in equation (13), for example). (But see Section 1.4.)
2. The weighting function will not have a common normalizing factor for all sampling position (as at the positions of the two vertical broken lines in Figure 1), and the interpolation process cannot be directly modeled as a convolution.

There is however one type of ISG that can be modeled and analyzed. It is the ISG that is a combination of relatively rotated and translated ESG. Because the basic unit is an ESG, then we don't have to worry about problem 1 above, but we will need to deal with the second one.

1.3.1 The combination of relatively rotated and translated ESG

To analyze this type of ISG we will need to switch from one to two dimensions. The Fourier transform pair now becomes

$$\begin{aligned} g(\mathbf{r}) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(\mathbf{w}) e^{j2\pi\mathbf{w}\cdot r} du dv \\ G(\mathbf{w}) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\mathbf{r}) e^{-j2\pi\mathbf{w}\cdot\mathbf{r}} dx dy, \end{aligned} \tag{34}$$

with $\mathbf{r}$ and $\mathbf{w}$ the position and frequency vectors, respectively. We will also need to use the following relation for the Fourier transform of a rotated map. More precisely, let $\mathbf{r}$ be defined with

$$\mathbf{r} = x\mathbf{e}_x + y\mathbf{e}_y, \tag{35}$$

then given a rotation matrix $\mathbf{R}$ we obtain a new vector

$$\mathbf{r}' = \mathbf{R}\mathbf{r} = x'\mathbf{e}_x + y'\mathbf{e}_y. \tag{36}$$

Applying this rotation to the map $g(\mathbf{r})$, we obtain a new map $g(\mathbf{R}\mathbf{r})$ with the corresponding Fourier transform

$$G'(\mathbf{w}) = \iint g(\mathbf{R}\mathbf{r}) e^{-j2\pi\mathbf{w}\cdot\mathbf{r}} dx dy. \tag{37}$$

We now make the change of variables $\mathbf{r}' = \mathbf{R}\mathbf{r}$ to get

$$G'(\mathbf{w}) = \iint g(\mathbf{r}') e^{-j2\pi\mathbf{w}\cdot\left(\mathbf{R}^{-1}\mathbf{r}'\right)} dx' dy' = \iint g(\mathbf{r}') e^{-j2\pi(\mathbf{R}\mathbf{w})\cdot\mathbf{r}'} dx' dy', \tag{38}$$

where the equation on the right hand side arises because the inverse matrix equals the transpose matrix for a rotation (i.e., $\mathbf{R}^{-1} = \mathbf{R}^{\mathrm{T}}$). We therefore find that if $g(\mathbf{r})$ has a Fourier transform $G(\mathbf{w})$, or

$$g(\mathbf{r}) \Leftrightarrow G(\mathbf{w}), \tag{39}$$

then

$$g(\mathbf{Rr}) \Leftrightarrow G(\mathbf{Rw}). \tag{40}$$

That is, we have the intuitive result that *the Fourier transform of a rotated map is the rotated-Fourier transform of the original map*. This result will greatly simplify our analysis.

Let's now consider a Dirac grid (which is just a generalization of the one-dimensional Dirac train) and its Fourier transform

$$\sum_{i,k=-\infty}^{\infty} \delta(x - il_1)\delta(y - kl_2) \Leftrightarrow \frac{1}{l_1 l_2} \sum_{m,n=-\infty}^{\infty} \delta\left(u - \frac{m}{l_1}\right)\delta\left(v - \frac{n}{l_2}\right). \tag{41}$$

From now on I will use the notation

$$\begin{aligned} \delta(\mathbf{r} - \mathbf{r}_{ik}) &\equiv \delta(x - il_1)\delta(y - kl_2) \\ \delta(\mathbf{w} - \mathbf{w}_{mn}) &\equiv \delta\left(u - \frac{m}{l_1}\right)\delta\left(v - \frac{n}{l_2}\right), \end{aligned} \tag{42}$$

with

$$\begin{aligned} \mathbf{r}_{ik} &= il_1\mathbf{e}_x + kl_2\mathbf{e}_y \\ \mathbf{w}_{mn} &= \frac{m}{l_1}\mathbf{e}_x + \frac{n}{l_2}\mathbf{e}_y, \end{aligned} \tag{43}$$

such that relation (41) becomes

$$\sum_{i,k=-\infty}^{\infty} \delta(\mathbf{r} - \mathbf{r}_{ik}) \Leftrightarrow \frac{1}{l_1 l_2} \sum_{m,n=-\infty}^{\infty} \delta(\mathbf{w} - \mathbf{w}_{mn}). \tag{44}$$

We now define an ISG $t'(\mathbf{r})$ composed of g rotated and transposed ESG as

$$t'(\mathbf{r}) = t(\mathbf{r}) \cdot \sum_{p=1}^{g} \sum_{i,k=-\infty}^{\infty} \delta(\mathbf{r} - \mathbf{R}_p\mathbf{r}_{ik} - \mathbf{a}_p), \tag{45}$$

where $\mathbf{R}_p$ and $\mathbf{a}_p = x_p\mathbf{e}_x + y_p\mathbf{e}_y$ are, respectively, the rotation matrix and the translation vector corresponding to grid p. As usual, we also have

$$t(\mathbf{r}) = \left[b(\mathbf{r}) * s(\mathbf{r})\right] * p(\mathbf{r}). \tag{46}$$

Using equations (40) and (44), it is straightforward to calculate the Fourier transform $T'(\mathbf{w})$ to be

$$\begin{aligned} T'(\mathbf{w}) &= T(\mathbf{w}) * \frac{1}{l_1 l_2} \sum_{p=1}^{g} \sum_{m,n=-\infty}^{\infty} e^{-j2\pi \mathbf{w}_{mn} \cdot \mathbf{a}_p} \delta\left(\mathbf{w} - \mathbf{R}_p \mathbf{w}_{mn}\right) \\ &= \frac{1}{l_1 l_2} \sum_{p=1}^{g} \sum_{m,n=-\infty}^{\infty} e^{-j2\pi \mathbf{w}_{mn} \cdot \mathbf{a}_p} T\left(\mathbf{w} - \mathbf{R}_p \mathbf{w}_{mn}\right). \end{aligned} \tag{47}$$

Presumably, our goal is to generate an ESG of steps $l_1/2$ and $l_2/2$ (along the x and y axes, respectively) from $t'(\mathbf{r})$. We call this new grid $t_3(\mathbf{r})$. Of course it will be necessary to use a weighting function $w(\mathbf{r})$ (which we choose to be normalized), so we may be tempted to assume that the interpolation process will simply consist of a convolution of $w(\mathbf{r})$ with $t'(\mathbf{r})$ followed by the appropriate re-sampling. But that would be an incorrect assumption. The reason for this can be understood from Figure 2.

In this figure, I have combined two ESG: one rotated by 10 degrees and another not rotated with respect to the coordinate axes. Both grids are not translated (see the caption). As can be seen, the combination of the two ESG produces an irregular pattern of Dirac functions, such that a predetermined weighting function can, at different locations, cover a different number of sampling functions at different relative positions from one another. The consequence of this will be the absence of a common normalizing factor at the different locations where an interpolation is done. If we denote this position-dependent normalizing factor with $n(\mathbf{r})$, then (from equation (30))

$$n(\mathbf{r}) = \left[\sum_q w\left(\mathbf{r} - \mathbf{r}_q\right)\right]^{-1}. \tag{48}$$

It should be clear from this that it would be inappropriate to assume that the first step of the interpolation process corresponds to a convolution with $w(\mathbf{r})$, since its effective amplitude changes with position according to equation (48). However, it is perfectly adequate to model the interpolated ESG $t_3(\mathbf{r})$ as follows

$$\begin{aligned} t_3(\mathbf{r}) &= \left[t'(\mathbf{r}) * w(\mathbf{r})\right] \cdot n(\mathbf{r}) \sum_{i,k=-\infty}^{\infty} \delta\left(\mathbf{r} - \frac{\mathbf{r}_{ik}}{2}\right) \\ &= \left\{\left[t(\mathbf{r}) \cdot \sum_{p=1}^{g} \sum_{s,t=-\infty}^{\infty} \delta\left(\mathbf{r} - \mathbf{R}_p \mathbf{r}_{st} - \mathbf{a}_p\right)\right] * w(\mathbf{r})\right\} \cdot n(\mathbf{r}) \sum_{i,k=-\infty}^{\infty} \delta\left(\mathbf{r} - \frac{\mathbf{r}_{ik}}{2}\right). \end{aligned} \tag{49}$$

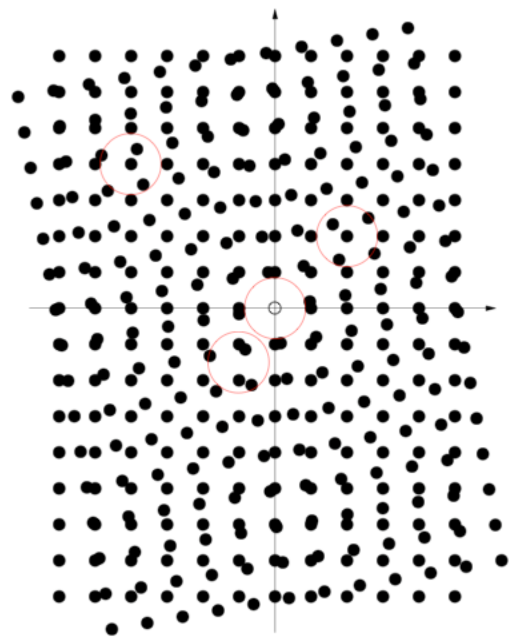

Figure 2 – A combination of two ESG. Every small dark circle corresponds to a Dirac function, and the position of the small empty circle is the origin of the maps, and of rotation for one of the two grids. Its rotation is of 10 degrees with respect to the coordinate axes, while the other grid is not rotated. Both ESG are not translated. The four bigger red empty circles correspond to the footprint of a predetermined weighting function.

That is, to mimic the position dependency of the normalization process, we simply have to multiply our initial ISG $t'(\mathbf{r})$ with the normalization function $n(\mathbf{r})$ *after* convolving with the weighting function.

On the other hand, we have to evaluate the effect this has on the determination of the proper weighting function to be used. To do so, we first calculate the Fourier transform of equation (49) to get

$$
\begin{aligned}
T_3(\mathbf{w}) &= \left(\left\{ \left[T(\mathbf{w}) * \frac{1}{l_1 l_2} \sum_{p=1}^{g} \sum_{q,r=-\infty}^{\infty} e^{-j2\pi \mathbf{w}_{qr} \cdot \mathbf{a}_p} \delta\left(\mathbf{w} - \mathbf{R}_p \mathbf{w}_{qr}\right) \right] W(\mathbf{w}) \right\} * N(\mathbf{w}) \right) \\
&\quad * \frac{4}{l_1 l_2} \sum_{m,n=-\infty}^{\infty} \delta\left(\mathbf{w} - 2\mathbf{w}_{mn}\right) \\
&= \left(\left\{ \left[\frac{1}{l_1 l_2} \sum_{p=1}^{g} \sum_{q,r=-\infty}^{\infty} e^{-j2\pi \mathbf{w}_{qr} \cdot \mathbf{a}_p} T\left(\mathbf{w} - \mathbf{R}_p \mathbf{w}_{qr}\right) \right] W(\mathbf{w}) \right\} * N(\mathbf{w}) \right) \\
&\quad * \frac{4}{l_1 l_2} \sum_{m,n=-\infty}^{\infty} \delta\left(\mathbf{w} - 2\mathbf{w}_{mn}\right).
\end{aligned} \tag{50}
$$

As was done for the one-dimensional case, we assume that the frequency extent of the weighting function $W(\mathbf{w})$ is such that it filters out the higher components of the spectrum corresponding to terms contained within the curly braces. That is, we assume that the only significant contribution from the corresponding summation happens when $q = r = 0$. With this simplification equation (50) becomes

$$T_3(\mathbf{w}) = \frac{4g}{l_1^2 l_2^2} \sum_{m,n=-\infty}^{\infty} \left\{ \left[T(\mathbf{w}) W(\mathbf{w}) \right] * N(\mathbf{w}) \right\} \Big|_{\mathbf{w}=\mathbf{w}-2\mathbf{w}_{mn}}, \tag{51}$$

and

$$t_3(\mathbf{r}) = \left\{ \left[t(\mathbf{r}) * w(\mathbf{r}) \right] n(\mathbf{r}) \right\} \cdot \frac{g}{l_1 l_2} \sum_{i,k=-\infty}^{\infty} \delta\left(\mathbf{r} - \frac{\mathbf{r}_{ik}}{2} \right). \tag{52}$$

We therefore see that these relations are the two-dimensional generalization of equations (21) and (22), with the provision that we include a multiplication by $n(\mathbf{r})$. Since this multiplication by $n(\mathbf{r})$ corresponds to a convolution in the frequency domain, it becomes apparent that *the spectrum of* $t(\mathbf{r}) * w(\mathbf{r})$ *is broadened by the spectrum of the normalization function* $n(\mathbf{r})$. That is, the interpolation process alters the spectrum of an ISG map in a way that is not present in the case of a RSG (or an ESG). This fact can be traced back to the lack of a common normalization factor for the weighting function. This justifies the previous definition for the RSG (see the discussion in the vicinity of equation (29)).

Despite its effect on the spectrum of the re-sampled map, the presence of the normalization function does not affect the criterions previously introduced (on page 5) to facilitate the determination of the weighting function. Finally, it should be noted that given an ISG, the normalization function $n(\mathbf{r})$ can, in principle, easily be determined, and its effect quantified. One can hope that it is sufficiently slowly varying that the spectrum broadening it causes is minimal.

1.4 The Effect of Missing Pixels

Although I stated earlier that there is no obvious way of generally expressing the Fourier transform of any irregularly spaced Dirac train, it is possible to get a glimpse as to the effect of missing pixels have on the representation of the signal.

To do so, we will now use the fact that a map is composed of a finite number of samples. More precisely, if the size of the detector array is $N_1 l_1 \times N_2 l_2$, where the indices 1 and 2 correspond to the x and y directions, respectively, then we assume that the two-dimensional pattern of dimension $N_1 N_2$ will repeat infinitely in all directions. The underlying assumption is that in the end the array will be windowed by the appropriate rectangular function to leave only the actual detector. Using this approach we can express the sampled signal (before windowing) as

$$t'(\mathbf{r}) = t(\mathbf{r}) \cdot \sum_{i,k} \sum_{m,n=-\infty}^{\infty} \delta(\mathbf{r} - \mathbf{r}_{mn} - \mathbf{a}_{ik}), \tag{53}$$

with this time (take note of the period)

$$\begin{aligned} \mathbf{r}_{mn} &= mN_1 l_1 \mathbf{e}_x + nN_2 l_2 \mathbf{e}_y \\ \mathbf{a}_{ik} &= p_i l_1 \mathbf{e}_x + q_k l_2 \mathbf{e}_y. \end{aligned} \tag{54}$$

That is, the sampling process is represented by a combination of translated Dirac trains of periodicity $(N_1 l_1, N_2 l_2)$. This way, we can deal with any missing pixel by omitting the corresponding translated Dirac train in equation (53). We now calculate the Fourier transform of the combined Dirac trains to be

$$\sum_{i,k} \sum_{m',n'=-\infty}^{\infty} \delta(\mathbf{r} - \mathbf{r}_{m'n'} - \mathbf{a}_{ik}) \Leftrightarrow \left[\sum_{i,k} e^{-j2\pi \mathbf{w} \cdot \mathbf{a}_{ik}} \right] \cdot \frac{1}{N_1 N_2 l_1 l_2} \sum_{m,n=-\infty}^{\infty} \delta(\mathbf{w} - \mathbf{w}_{mn}), \tag{55}$$

with

$$\mathbf{w}_{mn} = \frac{m}{N_1 l_1} \mathbf{e}_x + \frac{n}{N_2 l_2} \mathbf{e}_y. \tag{56}$$

Please note that that the minimum separation between the Dirac functions in frequency space is $1/Nl$, where Nl is the greatest between $N_1 l_1$ and $N_2 l_2$. We rewrite the right hand side of equation (55) as follows

$$D(\mathbf{w}) = \frac{1}{l_1 l_2} \sum_{m,n=-\infty}^{\infty} E(\mathbf{w}_{mn}) \delta(\mathbf{w} - \mathbf{w}_{mn}), \tag{57}$$

with

$$E(\mathbf{w}_{mn}) = \frac{1}{N_1 N_2} \sum_{i,k} e^{-j2\pi \mathbf{w}_{mn} \cdot \mathbf{a}_{ik}}. \tag{58}$$

To see how equations (57) and (58) can help us understand the effect of missing pixels, let's consider the case where the M^2 pixels of the "top" corner are missing. In this case, we have from equation (58) (with $p_i = i$ and $q_k = k$ in the second of equations (54))

$$
\begin{aligned}
N_1N_2E(\mathbf{w}_{mn}) &= \sum_{i=0}^{N_1-1} e^{-j2\pi im/N_1} \cdot \sum_{k=0}^{N_2-M-1} e^{-j2\pi kn/N_2} + \sum_{i=0}^{N_1-M-1} e^{-j2\pi im/N_1} \cdot \sum_{k=N_2-M}^{N_2-1} e^{-j2\pi kn/N_2} \\
&= \frac{1-e^{-j2\pi m}}{1-e^{-j2\pi m/N_1}} \cdot \frac{1-e^{-j2\pi n(N_2-M)/N_2}}{1-e^{-j2\pi n/N_2}} \\
&\quad + \frac{1-e^{-j2\pi m(N_1-M)/N_1}}{1-e^{-j2\pi m/N_1}} \cdot \frac{e^{-j2\pi n(N_2-M)/N_2}-e^{-j2\pi n}}{1-e^{-j2\pi n/N_2}} \\
&= e^{j\pi n(M+1)/N_2}\left\{(-1)^{m+n} e^{j\pi m/N_1} \frac{\sin(\pi m)}{\sin(\pi m/N_1)} \frac{\sin[\pi n(N_2-M)/N_2]}{\sin(\pi n/N_2)}\right. \\
&\quad \left. +(-1)^m e^{j\pi m(M+1)/N_1} \frac{\sin[\pi m(N_1-M)/N_1]}{\sin(\pi m/N_1)} \frac{\sin[\pi n M/N_2]}{\sin(\pi n/N_2)}\right\}.
\end{aligned} \tag{59}
$$

It can be seen that if all the pixels are accounted for (i.e., $M=0$), then equation (59) becomes

$$
\begin{aligned}
E(\mathbf{w}_{mn}) &= \frac{1}{N_1N_2}\left[\frac{1-e^{-j2\pi m}}{1-e^{-j2\pi\frac{m}{N_1}}}\right]\left[\frac{1-e^{-j2\pi n}}{1-e^{-j2\pi\frac{n}{N_2}}}\right] \\
&= (-1)^{m+n} \frac{e^{j\pi\left(\frac{m}{N_1}+\frac{n}{N_2}\right)}}{N_1N_2} \frac{\sin(m\pi)}{\sin(\pi m/N_1)} \frac{\sin(n\pi)}{\sin(\pi n/N_2)},
\end{aligned} \tag{60}
$$

which reduces to

$$
E(\mathbf{w}_{mn}) = \begin{cases} 1, & m = m'N_1 \text{ and } n = n'N_2 \\ 0, & \text{elsewhere} \end{cases} \tag{61}
$$

with m' and n' integer numbers. That is to say, equation (57) then becomes

$$
D(\mathbf{w}) = \frac{1}{l_1l_2} \sum_{m,n=-\infty}^{\infty} \delta\left(\mathbf{w}-\mathbf{w}_{(N_1m)(N_2n)}\right), \tag{62}
$$

which is the same result as was obtained earlier with equation (44) for a full ESG.

Returning to equation (59), it is probably necessary to plot this function to assess the global effect of the missing pixels. But it should be clear that the intensity of $E(\mathbf{w}_{mn})$ is in general non-zero for all values of m and n in this case. In other words, by removing even only one pixel we went from a case where we only had Dirac functions at intervals of l_1^{-1} and l_2^{-1} to a situation where they are now spaced by intervals of only $(N_1l_1)^{-1}$ and

$\left(N_2 l_2\right)^{-1}$. It is important, however, to also quantify the relative magnitude of these Dirac functions. For example, we find that

$$\begin{aligned}\frac{\left|E\left(\mathbf{w}_{10}\right)\right|}{\left|E\left(\mathbf{w}_{00}\right)\right|} &= \frac{M\left|\sin\left[\pi\left(N_1 - M\right)/N_1\right]/\sin\left(\pi/N_1\right)\right|}{N_1 N_2 - M^2} \\ &\simeq \frac{M^2}{N_1 N_2 - M^2}, \qquad \text{for } M \ll N_1.\end{aligned} \tag{63}$$

This last relation yields the intuitive result that *when only a small number of pixels are missing the amount of contamination is approximately equal to the ratio of the number of missing pixels to that of good pixels.* For SHARP where $N_1 = N_2 = 12$, we find that one bad pixel will give a ratio of 1/143, etc.

Finally, we calculate from equations (53) and (57) the spectrum of the measured signal to be

$$T'(\mathbf{w}) = \frac{1}{l_1 l_2} \sum_{m,n=-\infty}^{\infty} E\left(\mathbf{w}_{mn}\right) T\left(\mathbf{w} - \mathbf{w}_{mn}\right), \tag{64}$$

with

$$\mathbf{w}_{mn} = \frac{m}{N_1 l_1}\mathbf{e}_x + \frac{n}{N_2 l_2}. \tag{65}$$

Taking into account the windowing mentioned earlier, equation (64) transforms to

$$T'(\mathbf{w}) = N_1 N_2 \sum_{m,n=-\infty}^{\infty} \left\{E\left(\mathbf{w}_{mn}\right) T\left(\mathbf{w} - \mathbf{w}_{mn}\right) * \left[\operatorname{sinc}\left(\pi u' N_1 l_1\right) \operatorname{sinc}\left(\pi v' N_2 l_2\right)\right]\right\}. \tag{66}$$

But because of the narrowness of the sinc functions relative to that of $T(\mathbf{w})$, it is not expected that their presence will be of any significance.

It is apparent that missing pixels will bring some amount of aliasing that will be impossible to remove with a reasonably sized weighting function. The concept of Nyquist sampling can even lose much of its usefulness in a situation where too many pixels are missing. This fact strongly underlines the necessity of performing adequate dithers when observing with an imperfect detector array.