polarization measurement conventions for SHARP

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1. Measuring polarization on the celestial sphere

The convention used in astronomy is to give the *position angle* of the E-vector of the polarization. In astronomy, *position angle* is always zero for North and increases toward East. In other words, the polarization position angle (denoted by Φ_{sky}) has its zero for North-South polarization and increases counterclockwise as viewed in the sky (on the celestial sphere). Traditionally, the following definitions are used for the normalized Stokes' parameters q_{sky} and u_{sky} :

$$q_{sky} = (Q_{sky}/I) = P\cos(2\Phi_{sky})$$

$$u_{sky} = (U_{sky}/I) = P\sin(2\Phi_{sky}) \qquad (1)$$

From this we can derive the following table:

Table 1: polarization on the celestial sphere (a.k.a. "sky frame")

normalized Stokes' parameter	value of Φ_{sky}	orientation of E-vector
$+q_{sky}$	0°	N-S
$+u_{sky}$	45°	NE-SW
$-q_{sky}$	90°	E-W
$-u_{sky}$	135°	SE-NW

In physics (e.g., see Jackson's electromagnetism text) the convention is similar: As the incident wave moves through the sequence (+q) … (+u) … (-q) … (-u), the incident wave's E-vector rotates counterclockwise (CCW) as viewed by an observer looking at a wave moving toward that observer. In physics the zero-point is taken to be horizontal polarization rather than N-S polarization. We adopt the astronomers' convention here.

2. Measuring polarization in the azimuth-elevation reference frame

It is sometimes convenient to reference the measurements to the azimuth-elevation system. We adopt the same conventions as in the previous section, except that the reference direction is taken to be the vertical direction ("up"), rather than the N-S direction. The angle of polarization in this az-el frame will be denoted as $\Phi_{az\text{-}el}$.

Thus we have:

$q_{az\text{-}el} = P \cos(2\Phi_{az\text{-}el})$

$u_{az\text{-}el} = P \sin(2\Phi_{az\text{-}el})$ (2)

…from which we can derive the following table:

Table 2: polarization in the azimuth-elevation frame

normalized Stokes' parameter	value of $\Phi_{az\text{-}el}$	orientation of E-vector (as viewed looking outwards from telescope at sky)
$+q_{az\text{-}el}$	0°	vertical
$+ u_{az\text{-}el}$	45°	45° CCW from vertical
$- q_{az\text{-}el}$	90°	horizontal
$- u_{az\text{-}el}$	135°	45° CW from vertical

Now that we have defined two reference frames in which to measure polarization, we have to develop the formulas for moving back and forth between these frames. I.e., if we know $q_{az\text{-}el}$ and $u_{az\text{-}el}$, how do we derive q_{sky} and u_{sky}? One reason that we need to know this is that the instrumental polarization is tied to the instrument reference frame while the polarization of the astronomical source is tied to the sky (celestial) reference frame. To make things even more challenging, one component of the instrumetal polarization is induced by M3 which moves independently from the SHARP instrument. The next section deals with the formulas needed to move between reference frames and similar topics that will allow us to work through these issues.

<u>3. Rules for vector rotations and coordinate transformations</u>

<u>*vector rotations:*</u> Imagine that you have a vector **a** with Cartesian coordinates (a_x, a_y). If you rotate this vector in the *counterclockwise* direction by an angle ϕ then you get a different vector, call it **b**. Its Cartesian coordinates will be related to those of the original vector **a** by the following formula:

$$\begin{bmatrix} b_x \\ b_y \end{bmatrix} = \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix} \begin{bmatrix} a_x \\ a_y \end{bmatrix} \qquad (3)$$

To verify that I got the signs right, just let **a** = (1, 0) and rotate it by ϕ = 90°. According to the formula above, you get **b** = (0, 1), which we know to be the correct answer. Now try the same thing for $\phi = 270°$. You get $(0, -1)$. Once again, we know this to be the correct answer. So I put the minus sign in the right place in the matrix.

coordinate transformations: Imagine that you have a vector (call it **r**) and you measure its x-y coordinates (r_x, r_y) in the Cartesian coordinate system shown in blue below. Next imagine that you want measure its coordinates in a different Cartesian coordinate system which is related to the first one by a simple *counterclockwise* rotation by the angle ϕ, as shown in the diagram below. Let this new coordinate system be called the x'-y' coordinate system. It is shown in red.

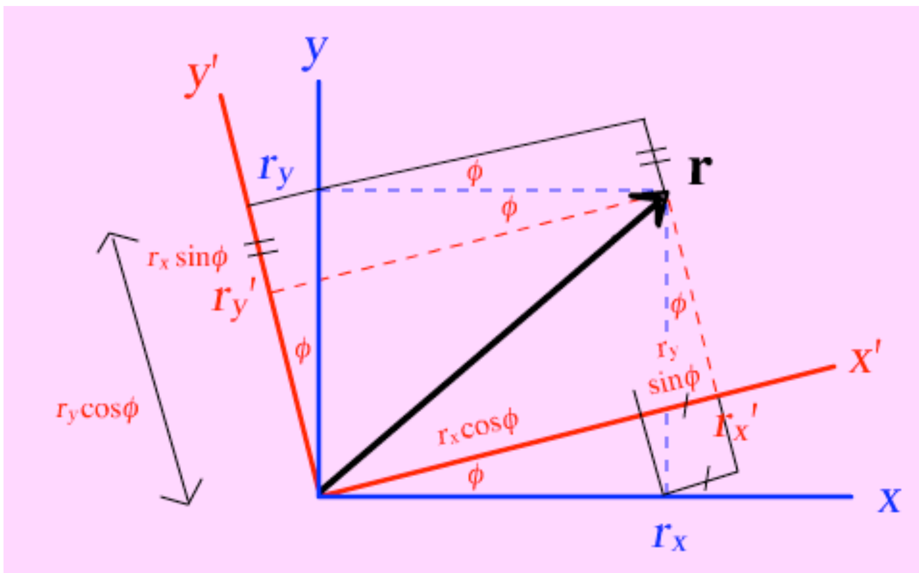

Figure 1: A vector **r** can be expressed as a pair of coordinates (r_x, r_y) in the blue Cartesian system, or alternatively as (r'_x, r'_y) in the red Cartesian system. (Figure taken from the Valdosta State University coordinate transformation tutorials.)

With a little trig you can show that the coordinates of **r** in the primed coordinate system are related to the coordinates of **r** in the unprimed system by a simple rotation matrix, as follows:

$$\begin{bmatrix} r'_x \\ r'_y \end{bmatrix} = \begin{bmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{bmatrix} \begin{bmatrix} r_x \\ r_y \end{bmatrix} \qquad (4)$$

Notice that the matrix in equation (4) is the inverse of the matrix in equation (3):

$$\begin{bmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{bmatrix} \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

…where I is the "identity matrix".

4. Relating the sky frame, the az-el frame, and the SHARP frame

When SHARP observes the sky, we need to know the relationship between "up" and "North on the sky". The parallactic angle α gives this information, according to the convention shown in Figure 2 below:

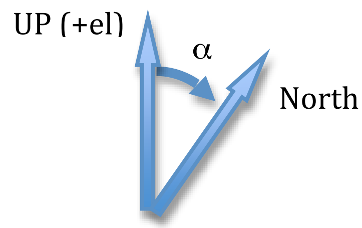

Figure 2: When looking out at the night sky, the celestial north direction (the direction given by moving along a great circle leading to the Pole star) makes some angle with respect to the "up" direction (the direction given by moving along a great circle leading to the zenith). That angle is referred to as the parallactic angle, α. As shown in the figure, α is the angle, measured clockwise, extending from "up" to "North".

Note that α is positive when "North" is *clockwise* with respect to "up". From the above diagram and the definitions and relationships given previously, we can see that

$$\Phi_{sky} = \Phi_{az\text{-}el} + \alpha \tag{5}$$

Also, using Figure 2 together with equation (4), we can see that

$$\begin{bmatrix} q_{az-el} \\ u_{az-el} \end{bmatrix} = \begin{bmatrix} \cos(2\alpha) & \sin(2\alpha) \\ -\sin(2\alpha) & \cos(2\alpha) \end{bmatrix} \begin{bmatrix} q_{sky} \\ u_{sky} \end{bmatrix} \tag{6a}$$

…where the factor of 2 in front of the α corresponds to the fact that a rotation by α in real space is equivalent to a rotation by 2α in Stokes' space.

Equation (6a) is equilavent to:

$$\begin{bmatrix} q_{sky} \\ u_{sky} \end{bmatrix} = \begin{bmatrix} \cos(2\alpha) & -\sin(2\alpha) \\ \sin(2\alpha) & \cos(2\alpha) \end{bmatrix} \begin{bmatrix} q_{az-el} \\ u_{az-el} \end{bmatrix} \tag{6b}$$

Next consider polarization measurements made in the reference frame of the SHARP instrument, which will be referred to as (q_{SH}, u_{SH}). It will probably be helpful to look at Figure 1 of Li et al. (2008) while reading the rest of this section. (This is the "Applied Optics" paper on SHARP.)

Let the SHARP reference frame be defined analogously to the az-el reference frame that is explained in Table 2. Thus we have $+q_{SH}$ representing vertical polarization as seen by someone looking from SHARP toward the tertiary mirror (M3), and $+u_{SH}$ representing radiation having its E-vector rotated by 45° in the counterclockwise direction, again, as seen by someone looking from SHARP toward M3.

Now imagine what Figure 1 of Li et al. (2008) would look like with the telescope pointing at the horizon. In this case a polarization vector that is vertical coming at the telescope ($+q_{az-el}$) will also be vertical as seen by SHARP ($+q_{SH}$). However, due to the reflection at M3, incoming $+u_{az-el}$ polarization will be seen by SHARP as $-u_{SH}$. Thus, for zero elevation, we have:

$$\begin{bmatrix} q_{SH} \\ u_{SH} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} q_{az-el} \\ u_{az-el} \end{bmatrix} \qquad \text{(for zero elevation)}$$

Now, what happens if we raise the telescope to a higher elevation? Well, as you increase the elevation, the stuff that someone looking from SHARP toward M3 sees will rotate counterclockwise. (Again, see Figure 1 of Li et al.) So we need to apply the a rotation matrix to the right side of the zero-elevation formula given above to get the *general* formula. From equation (3) we see that when you rotate a vector counterclockwise the rotation matrix has the minus sign in the upper right, so we obtain:

$$\begin{bmatrix} q_{SH} \\ u_{SH} \end{bmatrix} = \begin{bmatrix} \cos(2\varepsilon) & -\sin(2\varepsilon) \\ \sin(2\varepsilon) & \cos(2\varepsilon) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} q_{az-el} \\ u_{az-el} \end{bmatrix} \qquad (7)$$

…where ε stands for elevation and the factor of two multiplying ε reflects the fact that a rotation by ε in real space is a rotation by 2ε in Stokes' space, e.g. see equation (1).

<u>5. Relationships between bolometer output and polarization parameters</u>

SHARP collects data at four half-wave plate angles spaced by 22.5°, which we refer to as θ = 0°, θ = 22.5°, θ = 45°, and θ = 67.5°. The actual crystal fast and slow axes are at an unknown angle when the half-waveplate is at θ = 0°, since the half-wave plate is installed in an arbitrary fashion (no attempt to line up its fast and slow axes with any reference direction was made). Thus, one needs to calibrate the "zero angle" of the system. Recall also that the direction corresponding to positive θ was set arbitrarily. Thus, going from bolometer signals to (q_{SH}, u_{SH}) requires some careful consideration of signs, etc.

<u>*Case 1: Idealized system:*</u> We start with an idealized version of the SHARP polarimeter, to help clarify the situation. This is shown in Figure 3 below, where the idealized polarimeter is being viewed from above. The radiation proceeds from the *polarimeter input* through the HWP disk toward a polarizing grid that has its wires oriented horizontally. After passing through the grid the (now) vertically polarized radiation reaches the detector.

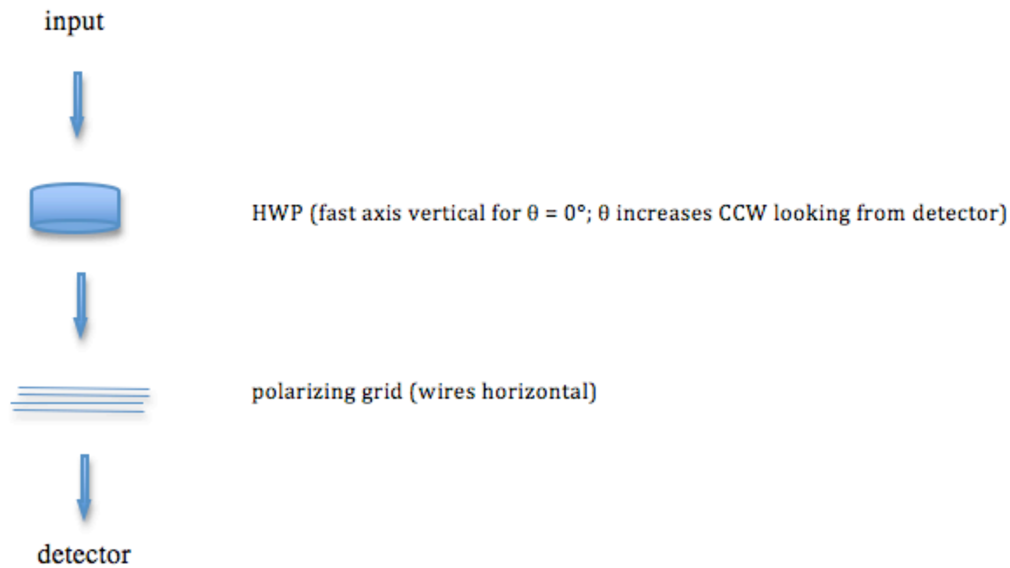

Figure 3: Idealized version of SHARP is shown above. The view shown here corresponds to looking down on the polarimeter from above. The input polarization (q_{SH}, u_{SH}) is measured by collecting data at four different halfwave-plate (HWP) angles, $\theta = 0°$, 22.5°, 45°, and 67.5°. The definition of θ for this idealized polarimeter is given in the figure.

Imagine that the HWP in our idealized polarimeter is initially set at $\theta = 0°$, which we will assume corresponds to fast and slow axes aligned vertically and horizontally, respectively. (In fact, it doesn't matter whether the it's the slow or the fast axis that's vertical, but to be definite I've made the fast axis vertical.) In this case, the detector will be most sensitive to *input polarization state* $+q_{SH}$. Now imagine that we wish to be most sensitive to $+u_{SH}$ (recall from Table 2 that this means input polarization rotated 45° *counterclockwise* from the vertical). To accomplish this, we need to rotate the HWP by 22.5° counterclockwise (CCW), so that the fast axis is 22.5° CCW from the vertical. In this case the $+u_{SH}$ (= 45° CCW) input polarization gets reflected about the fast axis and comes out as vertical polarization, to which our detector is most sensitive.

We can summarize the previous paragraph as follows: When starting with sensitivity to $+q_{SH}$, rotation of the HWP by 22.5° in the counterclockwise (CCW) direction will produce sensitivity to $+u_{SH}$.

Therefore, for the optical bench system described above, the measurement of (q_{SH}, u_{SH}) would be accomplished as follows:

$q_{SH} \propto$ signal(θ=0°) – signal(θ=45°)

$u_{SH} \propto$ signal(θ=22.5°) – signal(θ=67.5°) (8)

Where we have assumed that θ increases CCW as viewed *looking out from the detector*, as also noted in Figure 3.

Case 2: The real SHARP:

The idealized polarimeter described above has some similarities and some differences from SHARP.

One *similarity* is that θ does indeed *increase* as the HWP rotates CCW (as viewed looking out from the detectors). Specifically, the angular velocity vector corresponding to positive θ points away from M3 and toward SHARP, and the two reflections between the HWP and M3 (see Li et al. 2008) cancel so we need not consider their effects on the parity of the system.

One *difference* between the idealized system and SHARP is that (as noted above) the half-wave plate crystal axes are at an unknown angle when θ = 0°, which means that equations (8) are subject to rotation in (q_{SH}, u_{SH}) space by an arbitrary angle that must be determined experimentally ("zero angle" correction). Another difference has to do with the fact that, for historical reasons, the definitions of q and u used by SHARPINTEG2 do not match those given in equations (8). Instead, the following conventions are used by the SHARPINTEG2 code:

For *horizontally* polarized ("H") bolometers:

$q_{Raw} \propto$ signal(θ=0°) – signal(θ=45°)

$u_{Raw} \propto$ signal(θ=67.5°) – signal(θ=22.5°) (9a)

For *vertically* polarized ("V") bolometers:

$q_{Raw} \propto$ - {signal(θ=0°) – signal(θ=45°)}

$u_{Raw} \propto$ - {signal(θ=67.5°) – signal(θ=22.5°)} (9b)

To relate (q_{Raw}, u_{Raw}) to (q_{SH}, u_{SH}) we note the following three differences: (1) the arbitrary rotation discussed above, (2) a switch from V-positive (see figure 3) to H-positive (see equations 9) which amounts to a 90° rotation, (3) and a flip in the sign of u. To get from (q_{Raw}, u_{Raw}) to (q_{SH}, u_{SH}), therefore, we must apply a sign flip to u followed by a rotation by an arbitrary angle. (The 90° rotation gets swallowed up in the rotation by an arbitrary angle.) In practice, the arbitrary angle is determined experimentally by inserting light with known (q_{SH}, u_{SH}) and measuring (q_{Raw}, u_{Raw}). To restate all of this mathematically, we have:

$$\begin{bmatrix} q_{SH} \\ u_{SH} \end{bmatrix} = \begin{bmatrix} \cos 2\chi & \sin 2\chi \\ -\sin 2\chi & \cos 2\chi \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} q_{Raw} \\ u_{Raw} \end{bmatrix} \tag{10}$$

Where χ is arbitrary so the choice of whether to put the minus sign in the upper right or the lower left of the matrix is arbitrary, as is the use of the "2" in front of the χ. In any event, χ will be determined empirically as discussed above. For historical reasons, we use a variable called h to keep track of the arbitrary rotation, and we use the following equation to define h:

$$\begin{bmatrix} q_{SH} \\ u_{SH} \end{bmatrix} = \begin{bmatrix} \cos(2h + 180°) & \sin(2h + 180°) \\ -\sin(2h + 180°) & \cos(2h + 180°) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} q_{Raw} \\ u_{Raw} \end{bmatrix} \tag{11}$$

This is the same as equation (10) if you set $h = \chi - 90°$; we are of course free to use h instead of χ to keep track of the arbitrary rotation as long are we are always consistent about its definition.

Equation (11) can be simplified to obtain:

$$\begin{bmatrix} q_{SH} \\ u_{SH} \end{bmatrix} = \begin{bmatrix} -\cos(2h) & \sin(2h) \\ \sin(2h) & \cos(2h) \end{bmatrix} \begin{bmatrix} q_{Raw} \\ u_{Raw} \end{bmatrix} \tag{12}$$

…which we write as:

$$\begin{bmatrix} q_{SH} \\ u_{SH} \end{bmatrix} = R \begin{bmatrix} q_{Raw} \\ u_{Raw} \end{bmatrix} \tag{13}$$

…where we have introduced the matrix R (for "raw"), defined as:

$$R \equiv \begin{bmatrix} -\cos(2h) & \sin(2h) \\ \sin(2h) & \cos(2h) \end{bmatrix} \tag{14}$$

Since the matrix R is its own inverse (as can be easily verified by multiplying it by itself) we can also write:

$$\begin{bmatrix} q_{Raw} \\ u_{Raw} \end{bmatrix} = R \begin{bmatrix} q_{SH} \\ u_{SH} \end{bmatrix} \tag{15}$$

6. Application of the above equations: relating the raw and sky systems

Solving equation (7) for ($q_{az\text{-}el}$, $u_{az\text{-}el}$) and then combining the result with equations (6b) and (11), we obtain:

$$\begin{bmatrix} q_{sky} \\ u_{sky} \end{bmatrix} = \begin{bmatrix} \cos 2\alpha & -\sin 2\alpha \\ \sin 2\alpha & \cos 2\alpha \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos 2\varepsilon & \sin 2\varepsilon \\ -\sin 2\varepsilon & \cos 2\varepsilon \end{bmatrix} \begin{bmatrix} \cos(2h + \pi) & \sin(2h + \pi) \\ -\sin(2h + \pi) & \cos(2h + \pi) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} q_{Raw} \\ u_{Raw} \end{bmatrix}$$

…which is equivalent to:

$$\begin{bmatrix} q_{sky} \\ u_{sky} \end{bmatrix} = \begin{bmatrix} \cos 2\alpha & -\sin 2\alpha \\ \sin 2\alpha & \cos 2\alpha \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos 2\varepsilon & \sin 2\varepsilon \\ -\sin 2\varepsilon & \cos 2\varepsilon \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos(2h+\pi) & -\sin(2h+\pi) \\ \sin(2h+\pi) & \cos(2h+\pi) \end{bmatrix} \begin{bmatrix} q_{Raw} \\ u_{Raw} \end{bmatrix}$$

…which, in turn, is equivalent to:

$$\begin{bmatrix} q_{sky} \\ u_{sky} \end{bmatrix} = \begin{bmatrix} \cos 2\alpha & -\sin 2\alpha \\ \sin 2\alpha & \cos 2\alpha \end{bmatrix} \begin{bmatrix} \cos 2\varepsilon & -\sin 2\varepsilon \\ \sin 2\varepsilon & \cos 2\varepsilon \end{bmatrix} \begin{bmatrix} \cos(2h+\pi) & -\sin(2h+\pi) \\ \sin(2h+\pi) & \cos(2h+\pi) \end{bmatrix} \begin{bmatrix} q_{Raw} \\ u_{Raw} \end{bmatrix}$$

…and finally we obtain:

$$\begin{bmatrix} q_{sky} \\ u_{sky} \end{bmatrix} = \begin{bmatrix} \cos(2(\alpha+\varepsilon+h+90°)) & -\sin(2(\alpha+\varepsilon+h+90°)) \\ \sin(2(\alpha+\varepsilon+h+90°)) & \cos(2(\alpha+\varepsilon+h+90°)) \end{bmatrix} \begin{bmatrix} q_{Raw} \\ u_{Raw} \end{bmatrix} \tag{16}$$

This is equivalent to equations (10) and (11) of John's July 2007 memo which is posted on the analysis logbook.

<u>7. Application of the above equations: instrumental polarization correction</u>

As described in Li et al. (2008), the SHARP instrumental polarization (i.p.) has two components. One is fixed with respect to the SHARP instrument and here we express it in the *raw* frame as (q_i, u_i) where the "i" stands for instrument. The other is caused by M3 and is expressed in the *SHARP* frame as (q_T, u_T). Here "T" stands for telescope (M3 is part of the telescope). The vector (q_T, u_T) has the fixed value (q'_T, u'_T) when $\varepsilon = 0°$, but as ε is increased from zero, it is assumed to rotate CCW since the M3 mechanism rotates CCW in the SHARP frame as ε increases.

In the SHARP frame, the instrumental polarization is thus written:

$$\begin{bmatrix} q_{ip} \\ u_{ip} \end{bmatrix} = \begin{bmatrix} -\cos(2h) & \sin(2h) \\ \sin(2h) & \cos(2h) \end{bmatrix} \begin{bmatrix} q_i \\ u_i \end{bmatrix} + \begin{bmatrix} \cos(2\varepsilon) & -\sin(2\varepsilon) \\ \sin(2\varepsilon) & \cos(2\varepsilon) \end{bmatrix} \begin{bmatrix} q'_T \\ u'_T \end{bmatrix} \tag{17}$$

…where we have used equation (3) to determine that the negative sign should go in the upper right in the second matrix.

This equation is equivalent to equation (34) in John's July 2007 memo.

Note that we usually set $u'_T = 0$ and fit for q'_T which is expected on physical grounds to be positive (Li et al. 2008). Thus there are three free parameters in the i.p. fit: q_i , u_i , and q'_T . Once the i.p. is determined from a fit to the planet data, it can be subtracted from all observations. This subtraction is done in the program SHARPCOMBINE.