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A Proposal for Describing the SHARP Instrument Polarization

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Raw IP at Detector

Referenced to the difference of the H and V SHARP arrays (i.e. the *raw* Stokes parameters as defined in Giles 2006-Apr-13 memo) we can describe the instrument polarization (IP) as:

$$q = q_i + P_t \cos[-2(\epsilon - \delta)] \quad (1)$$

$$u = u_i + P_t \sin[-2(\epsilon - \delta)] \quad (2)$$

where the subscript i denotes the fixed part of the IP and the subscript t denotes the telescope part which varies with telescope elevation ϵ. The phase δ is dependent on the HWP zero angle and any non-zero phase introduced by the telescope IP itself.

These equations can be re-written

$$q = q_i + q_t \cos(2\epsilon) + u_t \sin(2\epsilon) \quad (3)$$

$$u = u_i - q_t \sin(2\epsilon) + u_t \cos(2\epsilon) \qquad \text{where} \quad (4)$$

$$q_t \equiv P_t \cos(2\delta) \quad (5)$$

$$u_t \equiv P_t \sin(2\delta). \quad (6)$$

Note that the expressions for q and u are symmetric when $\epsilon = 0$:

$$q = q_i + q_t \quad (7)$$

$$u = u_i + u_t. \quad (8)$$

In matrix form we can write

$$\begin{pmatrix} q \\ u \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cos(2\epsilon) & \sin(2\epsilon) \\ 0 & 1 & -\sin(2\epsilon) & \cos(2\epsilon) \end{pmatrix} \begin{pmatrix} q_i \\ u_i \\ q_t \\ u_t \end{pmatrix}. \quad (9)$$

This equation shows that the IP vector $(q_i\ u_i\ q_t\ u_t)^T$ can be determined by a simple linear regression if a set of raw stokes parameters (the LHS) are known for at least two elevation angles.

To compare the telescope IP at different values of the HWP zero-angle we need to rotate the IP. We project the instrument referenced Stokes parameters onto the sky (with angles measured east of north) using a standard rotation matrix.

$$\begin{pmatrix} q_{sky} \\ u_{sky} \end{pmatrix} = \begin{pmatrix} \cos(2\gamma) & -\sin(2\gamma) \\ \sin(2\gamma) & \cos(2\gamma) \end{pmatrix} \begin{pmatrix} q_{raw} \\ u_{raw} \end{pmatrix} \tag{10}$$

where

$$\gamma = \alpha + \epsilon + h + 90^\circ \tag{11}$$

where α is the parallactic angle, h is the HWP zero angle, and q_{raw} and u_{raw} are any instrument referenced Stokes parameters to be rotated to sky coordinates. The additional 90° is required to reference the sky angle east of north. To reference this to the location of M3 when the telescope is pointing at the horizon we set $\epsilon = \alpha = 0$. Note that we also need to correct for the reflection from M3 using the reflection matrix

$$\begin{pmatrix} q_3 \\ u_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} q_{sky} \\ u_{sky} \end{pmatrix}. \tag{12}$$

Note that

$$q_3 \equiv P_3 \cos(2\phi_3), \tag{13}$$

$$u_3 \equiv P_3 \sin(2\phi_3). \tag{14}$$

So to reference the telescope polarization at M3 we have

$$\begin{pmatrix} q_3 \\ u_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -\cos(2h) & \sin(2h) \\ -\sin(2h) & -\cos(2h) \end{pmatrix} \begin{pmatrix} q_t \\ u_t \end{pmatrix} \tag{15}$$

or

$$q_3 = -q_t \cos(2h) + u_t \sin(2h) \tag{16}$$

$$u_3 = q_t \sin(2h) + u_t \cos(2h) \tag{17}$$

So the telescope IP referenced at M3 is:

$$P_3^2 \equiv q_3^2 + u_3^2 \tag{18}$$

$$= q_t^2 + u_t^2 \tag{19}$$

$$P_3^2 = P_t^2 \quad \forall \quad h \tag{20}$$

and

$$\phi_3(h) \equiv \frac{1}{2}\arctan\left[\frac{u_3(h)}{q_3(h)}\right] \tag{21}$$

$$= \tfrac{1}{2}\arctan\{-\tan[2(\delta+h)]\} \tag{22}$$

$$\phi_3(h) = -\delta - h + n(90^\circ) \tag{23}$$

where the $n(90^\circ)$ term accounts for the 90° degeneracy of the $\tan(2\theta)$ function (n is any integer). To determine n we use the boundary values of:

$$q_t = 1, \quad u_t = 0 \quad \Rightarrow \quad \delta = 0 \quad \text{and} \quad \phi_3 = 90^\circ \tag{24}$$

where we have set $h = 0$ for simplicity. Therefore, we must have $n = 1$ in equation (23) and the final answer is

$$\phi_3(h) = -\delta - h + 90^\circ \tag{25}$$

The reader can confirm that this expression also holds for $q_t = -1$.

Of course the same steps can be followed to reference q_i and u_i at M3 rather than the array, yielding the same expression.

IP in front of SHARP

Given the rotation matrix R defined by equations (15)–(17) we can re-write equations (1) and (2) referenced between the entrance to SHARP and the teritary mirror M3 as:

$$R\begin{pmatrix} q \\ u \end{pmatrix} = R\begin{pmatrix} q_i \\ u_i \end{pmatrix} + R\begin{pmatrix} P_t\cos[-2(\epsilon-\delta)] \\ P_t\sin[-2(\epsilon-\delta)] \end{pmatrix} \tag{26}$$

where

$$R = \begin{pmatrix} -\cos(2h) & \sin(2h) \\ \sin(2h) & \cos(2h) \end{pmatrix}. \tag{27}$$

Substituting the quantity δ from equation (25) into equation (26) we have

$$R\begin{pmatrix} q \\ u \end{pmatrix} = \begin{pmatrix} q_i' \\ u_i' \end{pmatrix} + \begin{pmatrix} q_t'\cos(2\epsilon) - u_t'\sin(2\epsilon) \\ q_t'\sin(2\epsilon) + u_t'\cos(2\epsilon) \end{pmatrix} \tag{28}$$

where

$$q_i' = -q_i\cos(2h) + u_i\sin(2h) \tag{29}$$

$$u_i' = q_i\sin(2h) + u_i\cos(2h) \tag{30}$$

$$q_t' = P_t\cos(2\phi_3) \tag{31}$$

$$u_t' = P_t\sin(2\phi_3) \tag{32}$$

Notice that equations (29) – (32) are the same as equations (16) – (17) and (13) – (14).

Re-writing equation (28) in more convenient matrix notation we have

$$R\begin{pmatrix} q \\ u \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cos(2\epsilon) & -\sin(2\epsilon) \\ 0 & 1 & \sin(2\epsilon) & \cos(2\epsilon) \end{pmatrix} \begin{pmatrix} q_i' \\ u_i' \\ q_t' \\ u_t' \end{pmatrix}. \tag{33}$$

Or, using equations (29) – (30) for q_i' and u_i'

$$R\begin{pmatrix} q \\ u \end{pmatrix} = \begin{pmatrix} -\cos(2h) & \sin(2h) & \cos(2\epsilon) & -\sin(2\epsilon) \\ \sin(2h) & \cos(2h) & \sin(2\epsilon) & \cos(2\epsilon) \end{pmatrix} \begin{pmatrix} q_i \\ u_i \\ q_t' \\ u_t' \end{pmatrix}. \tag{34}$$

Note that $(q\ u)^T$ and $(q_i\ u_i)^T$ are *raw* Stokes parameters referenced at the detector array, while $(q_t'\ u_t')^T$ are referenced between the entrance of SHARP and M3.

The mixed formulation of equation (34) allows us to do two useful things. First, it may allow us to slightly break the inherent degeneracy between these 4 parameters. A degeneracy exists due to the limited range of the telescope elevation ($0° < \epsilon < 90°$). Since we are fairly certain that $\phi_3 = 0$ we can set $u_t' = 0$, leaving us with only 3 parameters. Second, since this formulation now depends on the HWP zero angle h, we can easily combine data sets from different observing runs with diffent values of h into one large data matrix which can be simultaneously inverted.